Journal of Engineering Physics and Thermophysics, Vol. 78, No. 1, 2005

FRACTIONAL INTEGRO-DIFFERENTIAL ANALYSIS OF HEAT AND MASS TRANSFER

L. P. Kholpanov and S. E. Zakiev UDC 66.02:621.1:533:51-74, 536.46; 532.5; 621.762

Application of the methods of fractional integro-differential analysis to an inhomogeneous canonical heat-conduction (diffusion) equation with inhomogeneous boundary conditions has enabled us for the first time to reduce the canonical heat-conduction equation to three equations of lower order that contain fractional-derivative operators. Examples and an analysis of those fundamental new possibilities that are opened up by such an approach to a wide class of problems of heat and mass exchange, combustion, self-propagating high-temperature synthesis, etc., have been given.

Introduction. Not only is the range of practical problems described with parabolic equations and systems vast for many chemical-technological and thermophysical processes, but it is also steadily extending. This is due to the fact that such processes are adequately described by diffusion and heat-transfer equations within the framework of modeling [1]. The practical interest expressed in these problems enabled one to accumulate an impressive arsenal of tools for investigating them by both analytical and numerical methods [2–5].

It is noteworthy that classification of the entire range of existing approaches is based on the well-known set of alternative techniques of analysis of the simplest problems of this class — canonical homogeneous and inhomogeneous heat-conduction equations [5]. The subject of this work is beyond the classification established, since we are dealing with a fundamentally new approach to solution of a classical problem of the form

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = f(x, t) , \quad t > 0 , \quad x > 0 ; \quad u(x, 0) = \psi(x) , \quad u(0, t) = \varphi(t) . \tag{1}$$

Since the method presented relies on apparatus commonly inapplicable to solution of classical parabolic problems, i.e., fractional integro-differential analysis [6–7], the present investigation can claim presentation of a new basic methodological idea. Not only is the result itself obtained in the work — equivalent reformulation (reduction) of Eq. (1) to these equations of lower order — completely new, but it is also unexpected. Thus, for example, the well-known reduction of a canonical hyperbolic equation, which underlies the characteristic method, reduces it to two equations of first order. Revealing such facts enables us to consider problems known for a long time from quite a new viewpoint, which is of particular interest for applied investigations.

It is noteworthy that one of the first attempts at applying fractional integro-differential analysis to parabolic models of chemical physics belongs to Yu. I. Babenko [8]. Focusing his attention just on the problem of description of flows at entry into a system with homogeneous initial conditions, Babenko demonstrated the possibilities of the original asymptotic approach created by him to a wide set of problems of practical importance. However, unfortunately, as he himself admits, his method was at least inconvenient for indirect asymptotic solution of parabolic problems even with homogeneous initial conditions, and the proposed technique of indirect calculation of flows at entry into the system relied purely on these conditions. It is conceivable that all this predetermined the rather cool attitude toward this attempt. We hope that the material presented in this work will enable one to consider this attempt in a somewhat different context.

Institute of Problems of Chemical Physics, Russian Academy of Sciences, Chernogolovka, Moscow Region, Russia; email: kholp@icp.ac.ru. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 78, No. 1, pp. 35–47, January–February, 2005. Original article submitted July 27, 2004.

Reduction of a Homogeneous Equation. Here we will consider the solution of a homogeneous equation, i.e., problem (1), on condition that $f(x, t) = 0$. For this purpose we will need determination of the fractional integral of order $\alpha > 0$ of the function f at $t > 0$ [6]

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{2}$$

and the Riemann–Liouville fractional derivative of order $m - 1 < \alpha < m$

$$\frac{d^{\alpha} f}{dt^{\alpha}}(t) = \frac{d^m}{dt^m} J^{m-\alpha} f(t) = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{f(\tau)\, d\tau}{(t-\tau)^{\alpha-m+1}}\right], & m-1 < \alpha < m\,; \\ \dfrac{d^m f}{dt^m}(t)\,, & \alpha = m\,. \end{cases} \tag{3}$$

In Laplace–Carson transformation [9], the transform of the original function $\dfrac{d^{\alpha} f}{dt^{\alpha}}(t)$ will be $p^{\alpha}\hat{f}(p)$ (where $\hat{f}(p)$ is the transform of $f(t)$), which enables us to pass from the formal determination of the operator

$$\exp\left(-\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x}{\sqrt{\alpha}}\right)^n \frac{\partial^{n/2}}{\partial t^{n/2}} \tag{4}$$

to its integral representation

$$\exp\left(-\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) f(x, t) \equiv \frac{\partial}{\partial t} \int_0^t \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha(t-\tau)}}\right) f(x, \tau)\, d\tau\,. \tag{5}$$

Its inverse operator will subsequently be needed only in the stage of intermediate calculations; therefore, we can reasonably content ourselves with its formal determination

$$\exp\left(\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x}{\sqrt{\alpha}}\right)^n \frac{\partial^{n/2}}{\partial t^{n/2}}\,. \tag{6}$$

From the independence of the variables t and x and immediately from determinations (4) and (6), it follows that both these operators commute with the operator $\partial^{1/2}/\partial t^{1/2}$ and possess regular differential properties of the exponent

$$\frac{\partial}{\partial x} \exp\left(-\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) = -\frac{1}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}} \exp\left(-\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right), \quad \frac{\partial}{\partial x} \exp\left(\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) = \frac{1}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}} \exp\left(\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right),$$

which, for example, immediately yield the relation

$$J^{1/2} \exp\left(-\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) = \frac{1}{\sqrt{\alpha}} \int_x^{\infty} \exp\left(-\frac{\xi}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) d\xi\,, \quad J^{1/2} \exp\left(\frac{x}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) = \frac{1}{\sqrt{\alpha}} \int_0^{x} \exp\left(\frac{\xi}{\sqrt{\alpha}} \frac{\partial^{1/2}}{\partial t^{1/2}}\right) d\xi\,. \tag{7}$$

Furthermore, subsequently in the work we will imply the following equalities:

$$\left(\frac{\partial}{\partial x}\right)^{-1} \ldots = \int_0^x \ldots \, dx \,, \tag{8}$$

$$\left(-\frac{\partial}{\partial x}\right)^{-1} \ldots = \int_x^\infty \ldots \, dx \,. \tag{9}$$

Finally, we need the operator

$$M(\psi)(t) = \frac{1}{2\sqrt{\pi\alpha t}} \int_0^\infty \exp\left(-\frac{\xi^2}{4\alpha t}\right) \psi(\xi) \, d\xi \,.$$

The main result of this section can be formulated as follows:

S t a t e m e n t 1. *The solution of the problem*

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \,, \quad t > 0 \,, \quad x > 0 \,; \quad u(x, 0) = \psi(x) \,, \quad u(0, t) = \varphi(t) + M(\psi)(t) \,, \quad t > 0 \,, \tag{10}$$

can be represented in the form of the sum

$$u = u_0 + u_1 + u_2 \,, \tag{11}$$

whose terms are the solutions of the following problems:

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \sqrt{\alpha} \frac{\partial}{\partial x}\right) u_0 = 0 \,, \quad u_0(x, 0) = 0 \,, \quad u_0(0, t) = \varphi(t) \,; \tag{12}$$

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \sqrt{\alpha} \frac{\partial}{\partial x}\right) u_1 = \frac{\psi(x)}{2\sqrt{\pi t}} \,, \quad u_1(x, 0) = \frac{\psi(x)}{2} \,, \quad u_1(0, t) = 0 \,, \quad t > 0 \,; \tag{13}$$

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \sqrt{\alpha} \frac{\partial}{\partial x}\right) u_2 = \frac{\psi(x)}{2\sqrt{\pi t}} \,, \quad u_2(x, 0) = \frac{\psi(x)}{2} \,, \quad u_2(0, t) = M(\psi)(t) \,, \quad t > 0 \,. \tag{14}$$

P r o o f. First of all, we note that, by virtue of the well-known theorem on the existence and uniqueness of the solution of (10), it is sufficient to show that u (11) is the solution of Eq. (10), since u determined from (11) satisfies the initial and boundary conditions of (10).

As is easy to see, the operator on the left-hand side of Eq. (10) can be represented in the form of the product of two commuting factors:

$$\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2} = \left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \sqrt{\alpha} \frac{\partial}{\partial x}\right)\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \sqrt{\alpha} \frac{\partial}{\partial x}\right). \tag{15}$$

Therefore, we have

$$\left(\frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2}\right)(u_0 + u_1 + u_2) = \left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \sqrt{\alpha} \frac{\partial}{\partial x}\right)\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \sqrt{\alpha} \frac{\partial}{\partial x}\right) u_0 +$$

$$+\left(\frac{\partial^{1/2}}{\partial t^{1/2}}-\sqrt{\alpha}\,\frac{\partial}{\partial x}\right)\left(\frac{\partial^{1/2}}{\partial t^{1/2}}+\sqrt{\alpha}\,\frac{\partial}{\partial x}\right)u_1+\left(\frac{\partial^{1/2}}{\partial t^{1/2}}+\sqrt{\alpha}\,\frac{\partial}{\partial x}\right)\left(\frac{\partial^{1/2}}{\partial t^{1/2}}-\sqrt{\alpha}\,\frac{\partial}{\partial x}\right)u_2=$$

$$=\left(\frac{\partial^{1/2}}{\partial t^{1/2}}-\sqrt{\alpha}\,\frac{\partial}{\partial x}\right)\frac{\psi(x)}{2\sqrt{\pi t}}+\left(\frac{\partial^{1/2}}{\partial t^{1/2}}+\sqrt{\alpha}\,\frac{\partial}{\partial x}\right)\frac{\psi(x)}{2\sqrt{\pi t}}=\frac{\partial^{1/2}}{\partial t^{1/2}}\left(\frac{\psi(x)}{\sqrt{\pi t}}\right)=0\,.$$

The above equality reflects the property of the fractional derivative to map a power function with a corresponding exponent into zero and not constants, as the ordinary derivative does [6]. The statement is proved.

Here two remarks must be made. First, we must recognize the fact that all the results of Statement 1 are a direct generalization to the case $\psi(x) = 0$ of [8], i.e., are true solely of that situation where (10) is equivalent to (12). Second, reduction of Statement 1 will be conjugate to the original approach to the solution of (10) (is not reduced, for example, to the operational method) only when the solutions of (12)–(14) are obtained on the basis of the procedure proposed (not used before). It is precisely such a method that is presented below; unlike the operational one, it is often called *operator* and is being actively developed now within the framework of "Idempotent Functional Analysis" [10] and is used in a number of cases for solution of nonlinear equations of fractional integro-differential analysis [11]. In this respect, in the present work, we propose a constructive approach to solution of a canonical parabolic equation describing a wide class of the processes of transfer of energy and a substance, whereas the operator method offers the most natural way of solving Eqs. (12)–(14) of fractional integro-differential analysis. Undoubtedly, the work does not seek to find a necessarily new approach to solution of (12)–(14). By virtue of the fact that, in these problems, we use the equations of fractional integro-differential analysis, it is precisely the operator method that offers the most natural way of solving them.

Solution of Reduction Problems for a Homogeneous Equation. We note that, by virtue of (7), we have a representation of one factor on the right-hand side of (15) in the form

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}}+\sqrt{\alpha}\,\frac{\partial}{\partial x}\right)u(x,t)=\sqrt{\alpha}\,\exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\frac{\partial}{\partial x}\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)u(x,t)\,, \tag{16}$$

which, with account for the representation (5), immediately yields that Eq. (12) is equivalent to the problem

$$\frac{\partial}{\partial x}\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)(u_0(x,t))=0\,,\quad u_0(x,0)=0\,,\quad u_0(0,t)=\varphi(t)\,. \tag{17}$$

Therefore, problem (12) in Statement 1 can generally be replaced by problem (17) with certainty. Since the operator $\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)$ becomes identical for $x = 0$, problem (17) is equivalent to

$$\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)u_0(x,t)=u_0(0,t)\quad\text{or}\quad\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)u_0(x,t)=\varphi(t)\,.$$

Finally, taking into account that $u_0(x, 0) = 0$, we arrive at the solution sought, formally applying the operator $\exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)$ to both sides of this equality:

$$u_0(x,t)=\exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\varphi(t)=\frac{\partial}{\partial t}\int_0^t\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha(t-\tau)}}\right)\varphi(\tau)\,d\tau\,. \tag{18}$$

Since we have erfc (0) = 1 and erfc (∞) = 0, for x = 0 the right-hand side of (18) exactly coincides with $\varphi(t)$, and for $x \neq 0$ we have

$$\frac{\partial}{\partial t}\int_0^t \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha(t-\tau)}}\right)\varphi(\tau)\,d\tau = \frac{x}{2\sqrt{\pi\alpha}}\int_0^t \frac{\exp\left(-\dfrac{x^2}{4\alpha(t-\tau)}\right)}{(t-\tau)^{3/2}}\varphi(\tau)\,d\tau \xrightarrow{t\to 0} 0\,.$$

We can also obtain the solution of (13) from the representation (16), constructing the operator inverse to the operator on the right-hand side of (16) and applying it to both sides of (13):

$$u_1(x,t) = \frac{1}{\sqrt{\alpha}}\exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\left(\frac{\partial}{\partial x}\right)^{-1}\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\left(\frac{\psi(x)}{2\sqrt{\pi t}}\right) =$$

$$= \exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\left(\frac{\partial}{\partial x}\right)^{-1}\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\frac{1}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\left(\frac{\psi(x)}{2}\right) =$$

$$= \exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\frac{1}{2\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\int_0^x \exp\left(\frac{\xi}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\psi(\xi)\,d\xi =$$

$$= \int_0^x \frac{1}{2\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\exp\left(-\left(\frac{x-\xi}{\sqrt{\alpha}}\right)\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\psi(\xi)\,d\xi = \frac{1}{2}\int_0^x \frac{\partial}{\partial \xi}\left(\exp\left(-\left(\frac{x-\xi}{\sqrt{\alpha}}\right)\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\right)\psi(\xi)\,d\xi =$$

$$= \frac{1}{2}\int_0^x \frac{\partial}{\partial x}\operatorname{erf}\left(\frac{x-\xi}{2\sqrt{\alpha t}}\right)\psi(\xi)\,d\xi = \frac{1}{2}\frac{\partial}{\partial x}\int_0^x \operatorname{erf}\left(\frac{x-\xi}{2\sqrt{\alpha t}}\right)\psi(\xi)\,d\xi\,.$$

Thus, we have

$$u_1(x,t) = \frac{1}{2}\frac{\partial}{\partial x}\int_0^x \operatorname{erf}\left(\frac{x-\xi}{2\sqrt{\alpha t}}\right)\psi(\xi)\,d\xi = \frac{1}{2\sqrt{\pi\alpha t}}\int_0^x \exp\left(-\frac{(x-\xi)^2}{4\alpha t}\right)\psi(\xi)\,d\xi\,. \tag{19}$$

Since

$$\lim_{s\to 0}\frac{1}{\sqrt{\pi s}}\exp\left(-\frac{x^2}{s}\right) = \delta(x)\,,$$

the right-hand side of (19) at $t \to 0$ represents a convolution of $\frac{1}{2}\delta(x)$ with $\psi(x)$; therefore, (19) satisfies the initial conditions of problem (13). At $t > 0$ and x = 0, the expression on the right-hand side of (19) vanishes, thus satisfying the boundary conditions of problem (13).

The solution of problem (14) relies on the representation

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \sqrt{\alpha}\frac{\partial}{\partial x}\right) = \sqrt{\alpha}\exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\frac{\partial}{\partial x}\exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)$$

and requires the same, in practice, calculations, as those in obtaining (19). In this case we arrive at the expression

$$u_2(x,t)=\frac{\psi(x)}{2}+\frac{1}{2}\frac{\partial}{\partial x}\int_x^\infty \operatorname{erfc}\left(\frac{\xi-x}{2\sqrt{\alpha t}}\right)\psi(\xi)\,d\xi=\frac{1}{2\sqrt{\pi\alpha t}}\int_x^\infty \exp\left(-\frac{(\xi-x)^2}{4\alpha t}\right)\psi(\xi)\,d\xi \qquad (20)$$

which satisfies the initial conditions of problem (14) at $t\to 0$ and takes the form

$$u_2(0,t)=\frac{1}{2\sqrt{\pi\alpha t}}\int_0^\infty \exp\left(-\frac{\xi^2}{4\alpha t}\right)\psi(\xi)\,d\xi=M(\psi)(t)$$

at $t>0$ and $x = 0$.

It has been mentioned above that the importance of the problem of expression of the substance flow modeled by problem (10) at entry into the system was demonstrated in [8]. As has been noted, this problem is of quite an independent interest in a number of cases of practical importance. Statement 1 enables us not only to solve this problem in the general case of inhomogeneous initial conditions but also to consider the general problem on the flow at any point of the system described by (10). Indeed, we have

$$-\alpha\frac{\partial}{\partial x}u(x,t)=-\alpha\frac{\partial}{\partial x}u_0(x,t)-\alpha\frac{\partial}{\partial x}u_1(x,t)-\alpha\frac{\partial}{\partial x}u_2(x,t)=\sqrt{\alpha}\,\frac{\partial^{1/2}}{\partial t^{1/2}}u_0(x,t)+\sqrt{\alpha}\,\frac{\partial^{1/2}}{\partial t^{1/2}}u_1(x,t)-$$

$$-\sqrt{\alpha}\,\frac{\psi(x)}{2\sqrt{\pi t}}-\sqrt{\alpha}\,\frac{\partial^{1/2}}{\partial t^{1/2}}u_2(x,t)+\sqrt{\alpha}\,\frac{\psi(x)}{2\sqrt{\pi t}}.$$

Thus, we arrive at the equality

$$-\alpha\frac{\partial}{\partial x}u(x,t)=\sqrt{\alpha}\,\frac{\partial^{1/2}}{\partial t^{1/2}}\left(u_0(x,t)+u_1(x,t)-u_2(x,t)\right), \qquad (21)$$

which yields the solution of the problem on the flow at entry into the system

$$-\alpha\frac{\partial}{\partial x}u(0,t)=\sqrt{\alpha}\,\frac{\partial^{1/2}}{\partial t^{1/2}}\left(\varphi(t)-M(\psi)(t)\right). \qquad (22)$$

In direct computations, representation of the second term on the right-hand side of (22) can turn out to be not entirely convenient; therefore, we express it in another manner, differentiating u_2 with respect to x and passing subsequently to the limit for $x\to\infty$:

$$-\alpha\frac{\partial}{\partial x}u_2(x,t)=-\alpha\frac{\partial}{\partial x}\frac{1}{2\sqrt{\pi\alpha t}}\int_x^\infty \exp\left(-\frac{(\xi-x)^2}{4\alpha t}\right)\psi(\xi)\,d\xi=$$

$$=\frac{\alpha}{2\sqrt{\pi\alpha t}}\frac{\partial}{\partial x}\int_x^\infty \exp\left(-\frac{(\xi-x)^2}{4\alpha t}\right)\psi(\xi)\,d\xi=\frac{\sqrt{\alpha}\,\psi(x)}{2\sqrt{\pi t}}-\frac{\sqrt{\alpha}}{2\sqrt{\pi t}}\int_x^\infty \frac{\partial}{\partial x}\left(\exp\left(-\frac{(\xi-x)^2}{4\alpha t}\right)\right)\psi(\xi)\,d\xi=$$

$$=\frac{\sqrt{\alpha}\,\psi(x)}{2\sqrt{\pi t}}+\frac{\sqrt{\alpha}}{2\sqrt{\pi t}}\int_x^\infty \frac{\partial}{\partial \xi}\left(\exp\left(-\frac{(\xi-x)^2}{4\alpha t}\right)\right)\psi(\xi)\,d\xi=-\frac{\sqrt{\alpha}}{2\sqrt{\pi t}}\int_x^\infty \left(\exp\left(-\frac{(\xi-x)^2}{4\alpha t}\right)\right)\frac{\partial\psi}{\partial\xi}(\xi)\,d\xi,$$

which yields

$$-\alpha\frac{\partial}{\partial x}u_2(0,t)=-\frac{\sqrt{\alpha}}{2\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{\xi^2}{4\alpha t}\right)\frac{\partial\psi}{\partial\xi}(\xi)\,d\xi\,.$$

Thus, in direct computations, instead of (22) we can use the formula

$$-\alpha\frac{\partial}{\partial x}u(0,t)=\frac{\sqrt{\alpha}}{\sqrt{\pi}}\frac{d}{dt}\int_0^t\frac{\varphi(\tau)\,d\tau}{\sqrt{t-\tau}}-\frac{\sqrt{\alpha}}{2\sqrt{\pi t}}\int_0^\infty \exp\left(-\frac{\xi^2}{4\alpha t}\right)\frac{\partial\psi}{\partial\xi}(\xi)\,d\xi\,. \tag{23}$$

We note that (23) is generally not equivalent to (22), since it assumes diffentiability of the function ψ; however, the requirement of smoothness in the initial conditions in solving problems related to diffusion and heat transfer, as a rule, is not in any way a substantial limitation.

Reduction of an Inhomogeneous Equations. The result of this section can also be represented in the form of a statement.

S t a t e m e n t 2. *The solution of the problem*

$$\frac{\partial u}{\partial t}-\alpha\frac{\partial^2 u}{\partial x^2}=f(x,t)\,,\quad t>0\,,\quad x>0\,;\quad u(x,0)=\psi(x)\,,\quad u(0,t)=\varphi(t)+M(\psi)(t)+F(t)\,,\quad t>0\,, \tag{24}$$

can be represented in the form of the sum

$$u=u_0+u_1+u_2\,, \tag{25}$$

whose terms are the solutions of the following problems:

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}}+\sqrt{\alpha}\frac{\partial}{\partial x}\right)u_0=0\,,\quad u_0(x,0)=0\,,\quad u_0(0,t)=\varphi(t)\,; \tag{26}$$

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}}+\sqrt{\alpha}\frac{\partial}{\partial x}\right)u_1=\frac{J^{1/2}f(x,t)}{2}+\frac{\psi(x)}{2\sqrt{\pi t}}\,,\quad u_1(x,0)=\frac{\psi(x)}{2}\,,\quad u_1(0,t)=0\,,\quad t>0\,; \tag{27}$$

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}}-\sqrt{\alpha}\frac{\partial}{\partial x}\right)u_2=\frac{J^{1/2}f(x,t)}{2}+\frac{\psi(x)}{2\sqrt{\pi t}}\,,\quad u_2(x,0)=\frac{\psi(x)}{2}\,,\quad u_2(0,t)=M(\psi)(t)+F(t)\,,\quad t>0\,, \tag{28}$$

where

$$F(t)=\frac{1}{\sqrt{\pi}}\int_0^\infty\int_0^t\frac{\exp\left(-\frac{\xi^2}{4\alpha(t-\tau)}\right)}{2\sqrt{t-\tau}}f(\xi,\tau)\,d\tau d\xi\,.$$

P r o o f. The proof reiterates, in fact, the proof of Statement 1 with account taken of the fact that

$$\frac{\partial^{1/2}}{\partial t^{1/2}}J^{1/2}=I\,.$$

In this section, we also find the solutions of (26)–(28). Since (26) exactly coincides with (12), the solution of this problem is yielded by formula (18). The solutions of (27) and (28) contain, unlike (19) and (20), one additional term each; let us compute these terms:

$$\frac{1}{\sqrt{\alpha}} \exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\left(\frac{\partial}{\partial x}\right)^{-1} \exp\left(\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\left(\frac{J^{1/2}f(x,t)}{2}\right) = \frac{1}{2\sqrt{\alpha}}\int_0^x \exp\left(\frac{(x-\xi)}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right) J^{1/2} f(\xi,t)\, d\xi =$$

$$= \frac{1}{2\sqrt{\alpha}}\int_0^x \frac{\partial}{\partial t}\int_0^t \operatorname{erfc}\left(\frac{x-\xi}{2\sqrt{\alpha(t-\tau)}}\right) J^{1/2} f(\xi,\tau)\, d\tau d\xi =$$

$$= \frac{1}{2}\exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\int_0^x \exp\left(\frac{\xi}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\frac{1}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}} Jf(\xi,t)\, d\xi =$$

$$= \frac{1}{2}\exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\int_0^x \frac{\partial \exp\left(\frac{\xi}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)}{\partial \xi} Jf(\xi,t)\, d\xi =$$

$$= \frac{1}{2}\left(Jf(x,t) - \exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right) Jf(0,t) - \int_0^x \exp\left(-\frac{(x-\xi)}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right)\frac{\partial Jf(\xi,t)}{\partial \xi}\, d\xi\right) =$$

$$= \frac{1}{2}\left(Jf(x,t) - \exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right) Jf(0,t) - \int_0^x \frac{\partial}{\partial t}\int_0^t \operatorname{erfc}\left(\frac{x-\xi}{2\sqrt{\alpha(t-\tau)}}\right)\frac{\partial Jf(\xi,\tau)}{\partial \xi}\, d\tau d\xi\right) =$$

$$= \frac{1}{2}\left(Jf(x,t) - \exp\left(-\frac{x}{\sqrt{\alpha}}\frac{\partial^{1/2}}{\partial t^{1/2}}\right) Jf(0,t) - \int_0^x\int_0^t \operatorname{erfc}\left(\frac{x-\xi}{2\sqrt{\alpha(t-\tau)}}\right)\frac{\partial f(\xi,\tau)}{\partial \xi}\, d\tau d\xi\right) =$$

$$= \frac{1}{2}\int_0^t\int_0^x \frac{\partial}{\partial \xi}\left(\operatorname{erfc}\left(\frac{x-\xi}{2\sqrt{\alpha(t-\tau)}}\right)\right) f(\xi,\tau)\, d\xi d\tau = \frac{1}{2\sqrt{\pi\alpha}}\int_0^x\int_0^t \frac{\exp\left(-\frac{(x-\xi)^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} f(\xi,\tau)\, d\tau d\xi\,.$$

Thus, we have

$$u_1(x,t) = \frac{1}{2\sqrt{\pi\alpha t}}\int_0^x \exp\left(-\frac{(x-\xi)^2}{4\alpha t}\right)\psi(\xi)\, d\xi + \frac{1}{2\sqrt{\pi\alpha}}\int_0^x\int_0^t \frac{\exp\left(-\frac{(x-\xi)^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} f(\xi,\tau)\, d\tau d\xi\,. \tag{29}$$

Analogously we arrive at the expression

$$u_2(x,t) = \frac{1}{2\sqrt{\pi\alpha t}}\int_x^\infty \exp\left(-\frac{(\xi-x)^2}{4\alpha t}\right)\psi(\xi)\, d\xi + \frac{1}{2\sqrt{\pi\alpha}}\int_x^\infty\int_0^t \frac{\exp\left(-\frac{(x-\xi)^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} f(\xi,\tau)\, d\tau d\xi\,. \tag{30}$$

We note that the right-hand side of (29) satisfies boundary conditions (27), whereas the right-hand side of (30) satisfies boundary conditions (28).

The entire general formula of the relationship between the gradient and the fractional derivatives (22) holds for the inhomogeneous equation, too. Its particular case — the flow at entry into the system — for an inhomogeneous equation takes the form

$$-\alpha \frac{\partial}{\partial x} u(0, t) = \sqrt{\alpha} \frac{\partial^{1/2}}{\partial t^{1/2}} (\varphi(t) - M(\psi)(t) - F(t)) . \quad (31)$$

Another form that is convenient for direct computations contains, in addition to the right-hand side of (23), the term

$$-\frac{1}{2\sqrt{\pi\alpha}} \lim_{x\to 0} \frac{\partial}{\partial x} \int_x^\infty \int_0^t \frac{\exp\left(-\frac{(x-\xi)^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} f(\xi, \tau)\, d\tau d\xi = \frac{1}{2\sqrt{\alpha}} J^{1/2} f(0, t) -$$

$$-\frac{1}{2\sqrt{\pi\alpha}} \lim_{x\to 0} \int_x^\infty \int_0^t \frac{\partial}{\partial x} \left(\frac{\exp\left(-\frac{(x-\xi)^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}}\right) f(\xi, \tau)\, d\tau d\xi = \frac{1}{2\sqrt{\alpha}} J^{1/2} f(0, t) +$$

$$+\frac{1}{2\sqrt{\pi\alpha}} \int_0^\infty \int_0^t \frac{\partial}{\partial \xi} \left(\frac{\exp\left(-\frac{\xi^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}}\right) f(\xi, \tau)\, d\tau d\xi = -\frac{1}{2\sqrt{\pi\alpha}} \int_0^\infty \int_0^t \frac{\exp\left(-\frac{\xi^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} \frac{\partial}{\partial \xi} f(\xi, \tau)\, d\tau d\xi .$$

Therefore, for the inhomogeneous equation we have

$$-\alpha \frac{\partial}{\partial x} u(0, t) = -\alpha \frac{\partial}{\partial x} u_{\text{hom}}(0, t) - \frac{1}{2\sqrt{\pi\alpha}} \int_0^\infty \int_0^t \frac{\exp\left(-\frac{\xi^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} \frac{\partial}{\partial \xi} f(\xi, \tau)\, d\tau d\xi , \quad (32)$$

where the first term on the right-hand side is replaced by the right-hand side of (23).

Discussion of the Results Obtained. The character of the behavior of the components of the solution of (10) can easily be illustrated. Figure 1 plots the solutions of problems (12)–(14) at different fixed instants of time as functions of space. The behavior of u_0 is typical of solutions of the general problem (10); therefore, it is not of interest for discussion. As follows from the plots presented in Fig. 1b, we have $u_1(x, t) \to \frac{1}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$ at $t \to \infty$, whereas $u_2(x, t) \to \frac{1}{2}$ (Fig. 1c). These results enable us, in analyzing the behavior of the solution of (10) with more complex functional dependences of φ and ψ for longer times ($t >> 0$), to restrict ourselves to selection of problems (12) and (13) or only of (12), carrying out the corresponding evaluations in advance. Figure 2, which demonstrates a much more rapid tendency than that of each individual term, is even more convincing of the efficiency of such an approach. Figure 3 gives the solution itself of problem (10).

$$u_1(x, t) + u_2(x, t) \to \frac{1}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{1}{2} ,$$

Passing to consideration of the example for an inhomogeneous problem, we note that Statement 2 can equivalently be reformulated for taking account of the well-known decomposition of this problem into the sum of the solution of a homogeneous problem and a certain particular solution of an inhomogeneous problem. Let us represent this reformulation.

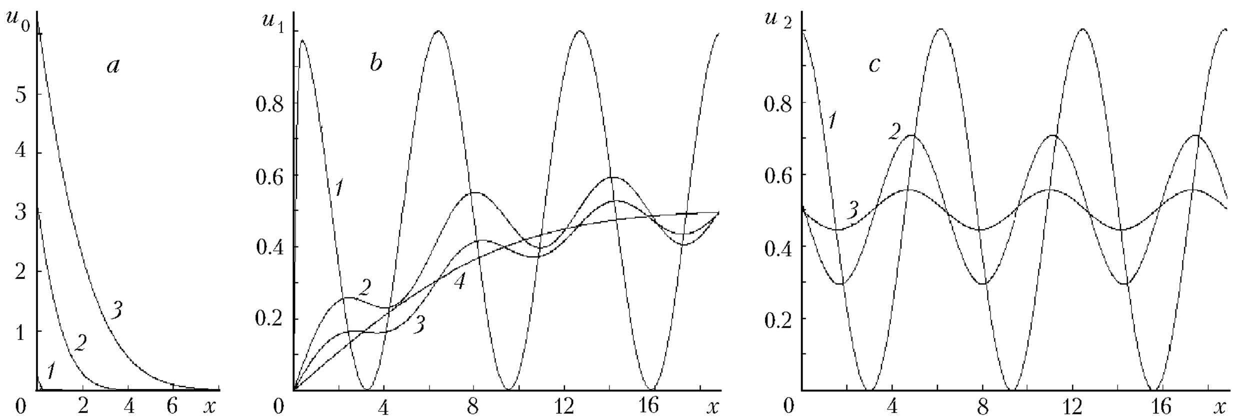
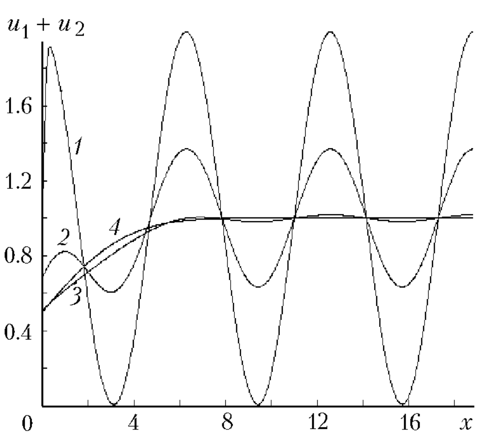


Fig. 1. Plots $u_0 = u_0(x, t_0)$ (solutions of problem (12)) (a), $u_1 = u_1(x, t_0)$ (solutions of problem (13)) (b), and $u_2 = u_2(x, t_0)$ (solutions of problem (14)) (c): a) at $t_0 = 0.1$ (1), 10 (2), and 40 (3) ($\alpha = 0.1$ and $\varphi(t) = \sqrt{t}$); b) at $t_0 = 0.1$ (1), 100 (2), and 270 (3); 4) $y = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t_0}}\right)$ at a fixed value of $t_0 = 270$ ($\alpha = 0.1$ and $\psi(x) = \cos(x) + 1$); c) at $t_0 = 0.1$ (1), 30 (2), and 270 (3) ($\alpha = 0.1$ and $\psi(x) = \cos(x) + 1$).



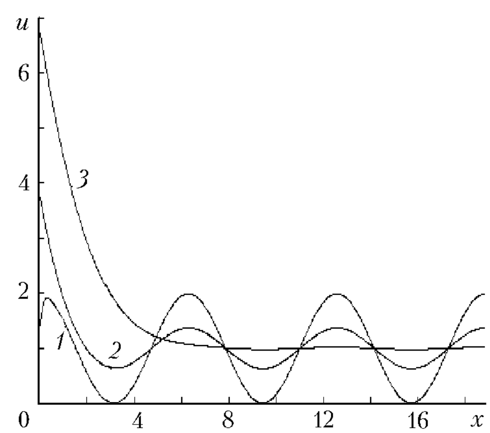


Fig. 2. Plots $u_1 + u_2 = u_1(x, t_0) + u_2(x, t_0)$: 1) at $t_0 = 0.1$, 2) 10, and 3) 40; 4) $y = \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t_0}}\right) + \frac{1}{2}$ at $t_0 = 40$.

Fig. 3. Plots $u(x, t) = u_0(x, t_0) + u_1(x, t_0) + u_2(x, t_0)$ (solutions of problem (10)): 1) at $t_0 = 0.1$, 2) 10, and 3) 40 ($\alpha = 0.1$, $\varphi(t) = \sqrt{t}$, and $\psi(x) = \cos(x) + 1$).

S t a t e m e n t 3. *The solution of the problem*

$$\frac{\partial \tilde{u}}{\partial t} - \alpha \frac{\partial^2 \tilde{u}}{\partial x^2} = f(x, t)\,, \quad t > 0\,, \quad x > 0\,, \quad \tilde{u}(x, 0) = \psi(x)\,, \quad \tilde{u}(0, t) = \varphi(t) + M(\psi)(t) + F(t)\,, \quad t > 0\,,$$

can be represented in the form of the sum

$$\tilde{u} = u + u_1^* + u_2^*\,, \tag{33}$$

where it is the solution of problem (10), whereas the remaining terms can be represented accordingly by the solutions of the problems

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} + \sqrt{\alpha}\,\frac{\partial}{\partial x}\right) u_1^* = \frac{J^{1/2} f(x, t)}{2}\,, \quad u_1^*(x, 0) = 0\,, \quad u_1^*(0, t) = 0\,, \quad t > 0\,, \tag{34}$$

$$\left(\frac{\partial^{1/2}}{\partial t^{1/2}} - \sqrt{\alpha}\,\frac{\partial}{\partial x}\right) u_2^* = \frac{J^{1/2} f(x, t)}{2}\,, \quad u_2^*(x, 0) = 0\,, \quad u_2^*(0, t) = F(t)\,, \quad t > 0\,, \tag{35}$$

where

$$F(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \int_0^t \frac{\exp\left(-\dfrac{\xi^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} f(\xi, \tau)\, d\tau d\xi\,.$$

Also, we note that, according to (29) and (30), we have the equalities

$$u_1^*(x, t) = \frac{1}{2\sqrt{\pi\alpha}} \int_0^x \int_0^t \frac{\exp\left(-\dfrac{(x-\xi)^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} f(\xi, \tau)\, d\tau d\xi\,, \tag{36}$$

$$u_2^*(x, t) = \frac{1}{2\sqrt{\pi\alpha}} \int_x^\infty \int_0^t \frac{\exp\left(-\dfrac{(x-\xi)^2}{4\alpha(t-\tau)}\right)}{\sqrt{t-\tau}} f(\xi, \tau)\, d\tau d\xi\,. \tag{37}$$

By virtue of Statement 3, it is sufficient to illustrate only solutions of the form (34) and (35). To consider the case of a traveling frontal wave (which is of interest for many problems of chemical physics) we took the right-hand side of Eq. (24) in the form

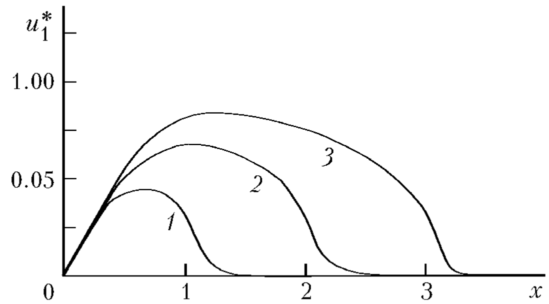
$$f(x, t) = \exp\left(-\frac{(x - vt + 2w)^2}{w}\right), \tag{38}$$

i.e., in that of a source moving with a constant velocity v and efficiently concentrated, at each fixed instant of time, only on a small segment whose size can be controlled with the constant w. Despite the framework of the linear formulation of the problem, such a form of the source is known to enable one to see, in solving it, many fundamental aspects of a more general situation — a quasistationary frontal wave that is the solution of a quasilinear problem. Since, for the homogeneous component of u, we can easily reformulate with (23) such requirements traditional for the formulation of the problem on a stationary wave as

$$\frac{\partial}{\partial x}\tilde{u}(0, t) \equiv 0\,,$$

in total accordance with problem (10), the process of its finding can no longer be considered.

The spatial distributions of u_1^*, resulting from (36) and presented in Fig. 4 at different fixed instants of time, point to the fact that this component of the general solution can be considered to be its dynamic part due to the presence of a moving source and moving in the same direction as the source does. As is easy to see from an analysis of the right-hand side of (36), the function $u_1^* = u_1^*(x, t_0)$ (as a function of x at any fixed instant of time t_0) will be dome-shaped — identical to that in Fig. 4 — for an extremely wide class of moving sources common to models of



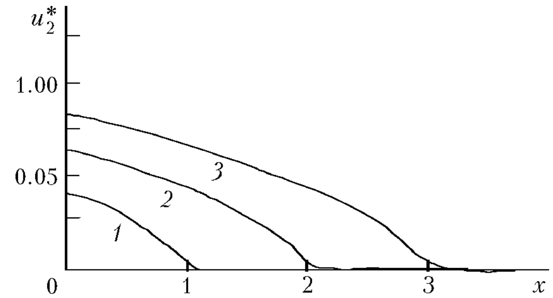


Fig. 4. Plots $u_1^* = u_1^*(x, t)$ (solutions of problem (34)): 1) at $t_0 = 1$, 2) 2, and 3) $3 \left(\alpha = 0.1 \text{ and } f(x, t) = \exp \left(-\left(\frac{x - vt + 2w}{w} \right)^2 \right) \right)$.

Fig. 5. Plots $u_1^* = u_1^*(x, t)$ (solutions of problem (35)): 1) at $t_0 = 1$, 2) 2, and 3) $3 \left(\alpha = 0.1 \text{ and } f(x, t) = \exp \left(-\left(\frac{x - vt + 2w}{w} \right)^2 \right) \right)$.

chemical physics and chemical engineering. In direct opposition to this, as Fig. 5 demonstrates, is the behavior of the component $u_2^* = u_2^*(x, t_0)$, which can be called the inertial part of the solution of the system due to the presence of a moving source. The above differentiation of u_1^* and u_2^* and their special features in accordance with the right-hand sides of (36) and (37) are fairly general in character; therefore, they enable us to give a formally rigorous definition of the velocity of the frontal thermal wave, namely, consider it to be coincident with the velocity of the maximum point of the space dome u_1^* at a given instant of time t_0. This can be formulated in a different manner as follows. If $h(t)$ is the function whose value coincides with the value of the variable x at the maximum point of $u_1^* = u_1^*(x, t)$ as a function of x at any fixed instant of time t, i.e., we have

$$\frac{\partial}{\partial x} u_1^* (h (t), t) = 0 , \tag{39}$$

then, according to the definition proposed, the instantaneous velocity $\omega(t)$ of the frontal wave at a given instant of time t should be considered to be

$$\omega (t) = \frac{dh}{dt} .$$

Even when this definition fails to be supported by specialists, the introduction of the function $h(t)$ implicitly prescribed by (39) is of interest, since it enables one to introduce the following new characteristic of a traveling frontal wave. Let us determine the function

$$U (t) \equiv u_1^* (h (t), t)$$

and note that (34) immediately yields

$$\left(\frac{\partial}{\partial t} + \sqrt{\alpha} \frac{\partial^{1/2}}{\partial t^{1/2}} \frac{\partial}{\partial x} \right) u_1^* = \frac{f (x, t)}{2} ,$$

whence, in accordance with (38), we obtain

$$\frac{\partial}{\partial t} U(t) = \frac{f(h(t), t)}{2} \equiv \Phi(t) \,. \tag{40}$$

Since the function $\Phi(t)$ can not infrequently be evaluated without knowing the direct form of the function $h(t)$ (for example, as a constant, a certain periodic function, etc.), the ordinary differential equation (40) resulting in this case can provide useful information even in a quasilinear situation where

$$\Phi(t) \equiv \frac{f(h(t), t, u_1^*(h(t), t))}{2} \,.$$

CONCLUSIONS

The fact that the potential of fractional integro-differential analysis has not been demonstrated up to the present time with the example of the most well-known and adequately studied problems of the type (1) interferes with its wider recognition as an efficient mathematical tool of modern applied investigations. The overwhelming part of the very limited number of its applications is constructed only on the basis of interpretations of a formal difference of a fractional derivative from an integral one and its unique possibilities of operation with such exotic objects as the Cantor staircase occurring in simulation models and statistical investigations [7]. As has already been noted above, work [8], even if it drops out of this list, is focused, nonetheless, on the particular problem of determination of the gradient at entry into the system, when neglect of all the problems indicated in Statements 1–3 except for (12) is justified. This is precisely the reason for the rigid limitation imposed there only on the case of zero initial conditions and for the tendentious declaration in [8] of the "physical absurdity" of the operator of problem (14). Not only do the statements formulated in the present work clearly point to the facts disregarded in those numerous examples [8], but they also open up the way for generalizations of the results obtained there to general cases of greater practical interest.

In the opinion of the authors, the work proposed can safely be considered as a presentation of the possibilities (remaining latent) of fractional integro-differential calculus for a substantial nontrivial analysis even of the most well-known classical problems. Moreover, the use of fractional integro-differential calculus turned out to involve at once a fundamentally new base for procedures of asymptotic and numerical calculations. Thus, solution (presented in the work) of the simple example of problem (10) quite clearly revealed that, in obtaining the approximation of the solution of a homogeneous problem at longer times, one can considerably reduce the volume of the required calculations, simplifying in the appropriate manner the form of the initial conditions in problems (13) and (14).

The given example of solution of problem (24) is even more impressive. The problem of formal determination of the velocity of a thermal wave in a frontally reacting system is among the fundamental problems involved in a vast number of problems of chemical physics and chemical engineering and has remained to be solved for more than half a century [12]. The way of solving it efficiently, directed in the present work in solution of the example of (24), seems not only natural but also the only one possible, particularly if we recall that a mere enumeration of works on this problem is much more vast than the volume of the present paper. Since, in the present investigation, the authors initially did not seek to study (for example, for quasilinear problems of macrokinetics [1]) this problem, the fact itself that an analysis of the simple example of (24) with the procedure presented in the work has brought us to a fundamentally new viewpoint of this important problem is very significant.

Fundamentally new opportunities opened up by the given statements and Eqs. (38) and (39) for analysis of an inverse problem are another aspect of interest, not touched upon directly in order to save space. The singled-out relations (32) substantially simplify an analysis of the boundary conditions. Composing u from the fragments singled out, we find ourselves in a situation most convenient for determination of the characteristics required. For example, the condition of nonincrease in the maximum of the function u_1^*, directly related to the properties of quasistationarity of the frontal solutions of (24), can quite efficiently be investigated with (34) and (39). At the same time, even the formulation of this condition itself in all the remaining approaches turns out to be very difficult.

Also noteworthy is the fact that we used operational calculus nowhere in the work (the right-hand side of (5) can be obtained directly from (4) after a series of somewhat cumbersome calculations). Primary emphasis was purposefully placed on direct operator methods, which have become an increasingly more efficient mathematical apparatus at present.

Summing up, we should note that the efficiency of application of fractional integro-differential analysis, which has the same age as the classical analysis [6], to applied investigations is not exhausted by the subjects covered above. The present work precisely instills confidence in the fact that applications of fractional integro-differential analysis that are of prime interest are to be discovered in the future.

NOTATION

In the work, we consider the problem in a dimensionless formulation; therefore, all the quantities given below are dimensionless.

$\left(\frac{\partial}{\partial x}\right)^{-1}$, operator determined by (8); $\left(-\frac{\partial}{\partial x}\right)^{-1}$, operator determined by (9); erf $(t) = \frac{2}{\sqrt{\pi}}\int_0^t \exp(-\tau^2)d\tau$; erfc $(t) = 1 - \text{erf}(t)$; $f(x, t)$, inhomogeneous part of the canonical heat-conduction equation; J, integral operator; m, natural number; I, identity operator; $h(t)$, solution of Eq. (39); t, variable; $u(x, t)$, unknown function in the homogeneous canonical heat-conduction equation; $\tilde{u}$, unknown function in the inhomogeneous canonical heat-conduction equation; u_0, solution of problem (12); u_1, solution of problem (13); u_2, solution of problem (14); u_1^*, solution of problem (34); u_2^*, solution of problem (35); v, velocity of the source (38); w, characteristic width of the source (38); x, variable; α, constant coefficient of the canonical heat-conduction equation of the second derivative with respect to x, fractional integration and differentiation order; $\delta(x)$, Dirac δ function; $\Gamma(\alpha)$, gamma function; ξ and τ, integration constants; $\varphi(t)$, boundary conditions in the canonical heat-conduction equation; $\psi(x)$, initial condition in the canonical heat-conduction equation; $\omega(t)$, instantaneous velocity of the frontal wave.

REFERENCES

1. D. A. Frank-Kamenetskii, *Diffusion and Heat Transfer in Chemical Kinetics* [in Russian], Nauka, Moscow (1987).
2. A. Friedman, *Partial Differential Equations of Parabolic Type* [Russian translation], Mir, Moscow (1968).
3. V. Ya. Arsenin, *Methods of Mathematical Physics and Special Functions*, [in Russian], Nauka, Moscow (1984).
4. Ş. G. Krein, *Linear Differential Equations in the Banach Space* [in Russian], Nauka, Moscow (1967).
5. É. M. Kartashov, *Analytical Methods in the Theory of Heat Conduction of Solids* [in Russian], Vysshaya Shkola, Moscow (2001).
6. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Integrals and Derivatives of Fractional Order and Some Applications of Them* [in Russian], Nauka i Tekhnika, Minsk (1987).
7. A. Carpinteri and F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien–New York (1997).
8. Yu. I. Babenko, *Heat and Mass Transfer. Method of Calculation of Thermal and Diffusion Flows* [in Russian], Khimiya, Leningrad (1986).
9. V. A. Ditkin and A. P. Prudnikov, *Operational Calculus* [in Russian], Vysshaya Shkola, Moscow (1966).
10. G. L. Litvinov, V. P. Maslov, and G. B. Shpiz, Idempotential functional analysis. Algebraic approach, *Mat. Zametki*, **69**, No. 5, 1–39 (2001).
11. G. Dattoli, P. E. Ricci, and D. Sacchetti, Generalized shift operators and pseudo-polynomials of fractional order, *Appl. Math. Comput.*, No. 141, 215–224 (2003).
12. Ya. B. Zel'dovich, G. I. Barenblat, V. B. Librovich, and G. M. Makhviladze, *Mathematical Theory of Combustion and Explosion* [in Russian], Nauka, Moscow (1980).